In summary, Eqs. (1) and (2) provide a quite good empirical model for the behavior of turbulent vortex rings and are valid to reasonably large distances from the discharging orifice. The equations also have the virtue of involving very small virtual times and distances and hence are not overly sensitive to inaccuracies in the determination of these quantities.

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Eddy Viscosity Distributions in a Mach 20 Turbulent Boundary Layer

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Nomenclature

A = damping scale, Eq. (4)
A^* = damping constant, Eq. (5)
C_f = skin friction coefficient
K = "Prandtl wall slope" on mixing length, Eq. (4)
l = Prandtl mixing length
M = Mach number
P = pressure
R_θ = Reynolds number based on compressible momentum thickness
r = radius
T_w = wall temperature
T_t = freestream stagnation temperature
u,v = longitudinal and normal velocity components, respectively
x,y = cartesian coordinates along and normal to the nozzle wall, respectively
δ^* = compressible displacement thickness
δ_i^* = incompressible displacement thickness
ρ = density
δ^+ = $\rho_w\delta(\tau_w/\rho_w)^{1/2}/\mu_w$
τ = shear stress
ϵ = eddy viscosity
μ = molecular viscosity
δ = density boundary-layer thickness, y at $\rho/\rho_e = 0.995$

Subscripts

e = local value external to boundary layer
w = wall value

Received December 3, 1970; revision received January 11, 1971.

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MOST current methods for computing compressible turbulent boundary-layers rely upon some assumed model of the turbulent shear or Reynolds stress term which appears in the momentum equation for the mean flow. Assumed models can be validated in two ways; one can either compute results for various test cases and compare these with experimental profiles, or one can work backward from the experimental data using the mean flow equations and directly compare the resulting shear stress and eddy viscosity distributions with the assumed model. The latter method has been utilized in several investigations[1–4] up to a Mach number of 7. The results of this work, particularly Refs. 2 and 4, greatly aided the development of computational techniques for compressible turbulent boundary layers. Recently, detailed nozzle boundary-layer profile data at several longitudinal stations became available in the Mach 20 range.[5] This Note presents shear stress and eddy viscosity distributions obtained from this high Mach number profile data and indicates the most consistent of the available models for turbulent shear.

The comparisons shown in Ref. 5 prompted this work by indicating that the method of Ref. 6 was unable to predict observed velocity profile development at Mach 20 (Fig. 6 of Ref. 5). This data provide a severe test of "compressibility effects" upon turbulent shear models because of the large density change (factor of 140) across the boundary layer. As the data of Ref. 5 were obtained on the inside wall of an axisymmetric nozzle, the following mean flow equations are used

Continuity

$$\partial(\rho ur)/\partial x + \partial(\rho vr)/\partial y = 0 \qquad (1)$$

Momentum

$$\partial(\rho u^2 r)/\partial x + \partial(\rho uvr)/\partial y = -r(dPe/dx) + \partial(r\tau)/\partial y \qquad (2)$$

where $r = r_w - y$ and $\tau = (\mu + \epsilon)(du/dy)$.

In the usual fashion an expression for ρvr obtained by integrating Eq. (1) is substituted into Eq. (2) and Eq. (2) integrated once with respect to y. Then, u/u_e and ρ/ρ_e are assumed to be functions of y/δ only[4] (a fairly accurate assumption for the data of Ref. 5).

The resulting equation follows:

$$\frac{\tau}{\rho_e u_e^2}\left(1 - \frac{y}{r_w}\right) = \frac{C_f}{2} + \int_0^{y/\delta} \frac{\rho}{\rho_e}\left(\frac{u}{u_e}\right)^2 dy/\delta \times \left[\frac{\delta}{\rho_e u_e^2}\frac{d(\rho_e u_e^2)}{dx} + \frac{d\delta}{dx} + \frac{\delta}{r_w}\frac{dr_w}{dx}\right] - \int_0^{y/\delta} y/\delta \times \frac{\rho}{\rho_e}\left(\frac{u}{u_e}\right)^2 dy/\delta \left[\frac{\delta^2}{r_w\rho_e u_e^2}\frac{d(\rho_e u_e^2)}{dx} + \frac{2\delta}{r_w}\frac{d\delta}{dx}\right] - \int_0^{y/\delta}\frac{\rho}{\rho_e}\frac{u}{u_e} dy/\delta\left[\frac{u}{u_e}\frac{\delta}{\rho_e u_e}\frac{d(\rho_e u_e)}{dx} + \frac{u}{u_e}\frac{d\delta}{dx} + \frac{u}{u_e}\frac{\delta}{r_w}\frac{dr_w}{dx}\right] + \int_0^{y/\delta} y/\delta\,\frac{\rho}{\rho_e}\frac{u}{u_e} dy/\delta\left[\frac{u}{u_e}\frac{\delta^2}{r_w\rho_e u_e}\frac{d(\rho_e u_e)}{dx} + \frac{u}{u_e}\frac{\delta 2}{r_w}\frac{d\delta}{dx}\right] + \frac{dP_e}{dx}\frac{\delta}{\rho_e u_e^2} y/\delta - \frac{dP_e}{dx}\frac{\delta^2}{r_w}\frac{(y/\delta)^2}{2}\frac{1}{\rho_e u_e^2} \qquad (3)$$

Equation (3) allows the determination of $\tau/\rho_e u_e^2$ as a function of y/δ once the various x derivatives and velocity and density profiles are known. Here profiles obtained at station 108 along the wall of the 22-in. Mach 20 helium tunnel are examined (taken from Table 2 of Ref. 5). At this station, small external pressure and wall radius gradients still exist. Input values used in the solution of Eq. (3) include $C_f/2 = 9.75 \times 10^{-5}$, $d\delta/dx = 0.056$, $dr_w/dx = 0.06$, $T_w/T_t = 1.0$ Figure 1 shows results of the calculation. The total shear has been separated into its laminar and turbulent components using the conventional definition of laminar shear. The δ used in the present work is the pitot or density thickness given in Ref. 5. The nominal velocity thickness is much less, as

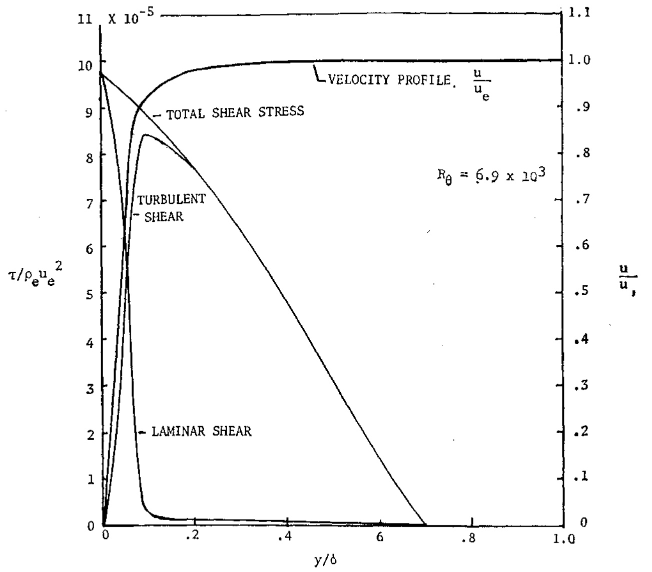

Fig. 1 Shear stress distributions, $M_e = 20.8$, $T_w/T_t \approx 1.0$, station 108 from data of Ref. 5.

indicated by the u/u_e profile also shown on the figure ($u/u_e = 0.995$ corresponds to $y/\delta = 0.5$). The computed shear stress becomes essentially zero where $u/u_e = 0.9983$ and $\rho/\rho_e = 0.71$. The fact that $\tau \approx 0$ at $y/\delta < 1$ may be reasonable in this case because of the extreme fullness of the velocity profile, i.e., $\partial(u/u_e)/\partial(y/\delta) \cong 0$ when $y/\delta < 1$. Also, the hot wire data of Ref. 5 for total temperature fluctuations are interpretable in the outer region as twice the velocity fluctuation level because of the high Mach number, and these results indicate that the longitudinal velocity fluctuation level is less than 0.5 percent for $y/\delta > 0.7$.

Using the turbulent shear stress distribution from Fig. 1, eddy viscosity and mixing length distributions were obtained as shown on Fig. 2. As indicated on Fig. 2, in the outer portion of the boundary layer the eddy viscosity and mixing length levels are significantly greater at Mach 20 than those obtained in Ref. 2 for $M \leq 5$. However, comparison of the present results with those of Ref. 7 on a δ^+ rather than an R_θ basis indicates that the large l/δ and $\epsilon/u_e\delta_i^*$ values found in the present case are due to a low Reynolds number effect previously found[7] at low speeds rather than any "Mach number" effect.

For the inner portion of the boundary layer the mixing length (shown on Fig. 2b) seems to provide a reasonable model for shear stress. However, the "Prandtl wall slope" on the l/δ vs y/δ variation is found to be closer to 0.6 than to the 0.4 variation obtained in Ref. 2 and used in Ref. 6. This increased slope of l with y constitutes the major reason for the disagreement between data and theory indicated in Ref. 5. Values of up to 0.6 for this Prandtl wall slope have recently been noted at low speeds by H. McDonald (private communication). Again, these larger values are evidently typical of low Reynolds number turbulent boundary layers.

Near the wall ($y/\delta \leq 0.1$), the l distribution shows evidence of being damped by the wall and in fact, exhibits a variation similar to that used by Van Driest to account for wall damping effects. The extension of the so-called "Van Driest damping function"[8] to compressible flows is currently an area of uncertainty. The basic expression[8] is

$$l/\delta = Ky/\delta(1 - e^{-y/A}) \tag{4}$$

where

$$A = A^*\mu/\rho(\tau/\rho)^{1/2} \tag{5}$$

and A^* is the "damping constant" usually taken to be 26 for impermeable wall flows.[6] In compressible flows, τ in Eq. (5)

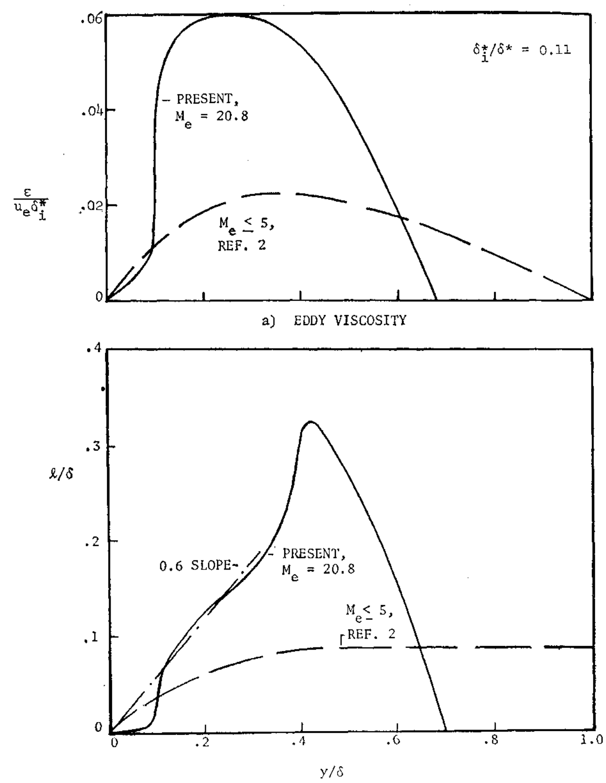

Fig. 2 Eddy viscosity and mixing length distributions, $T_w/T_T \approx 1.0$, $M_e = 20.8$, station 108 from data of Ref. 5.

is usually evaluated at the wall. This same approach was used in Ref. 6 for hypersonic flow where τ, ρ, and μ were evaluated at the wall. Patankar[9] indicates that the local τ should be used in Eq. (5) and Cebeci[10] suggests using the wall shear but averaging μ and ρ over the sublayer for compressible flows.

The results shown on Fig. 2b provide an opportunity to determine the more correct approach as almost a factor of 10 change in density occurs across the sublayer region. Therefore, using the l/δ variation shown in Fig. 2b and $K = 0.6$,

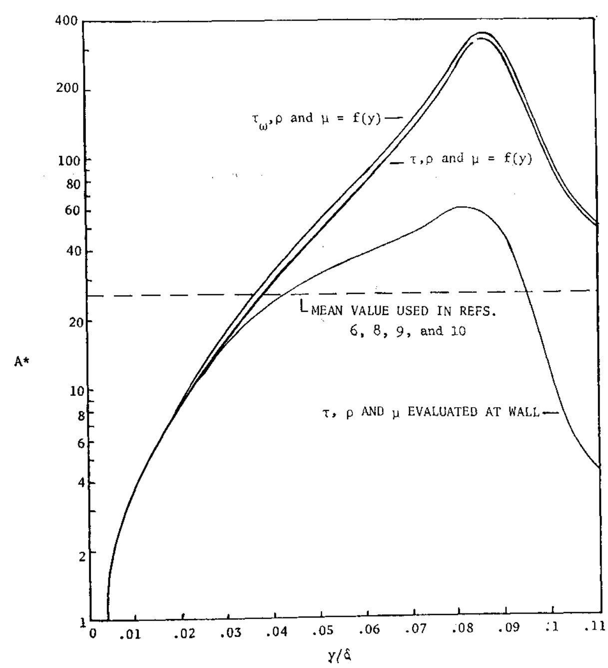

Fig. 3 Variation of damping constant, $M_e = 20.8$, using results of Fig. 2b.

values of A^* have been obtained as a function of y with various assumptions for τ, ρ, and μ in Eq. 5. If Eq. (4) is correct, A^* should be constant with respect to y. Figure 3 shows that large variations in A^* occur in the region $y/\delta \leq 0.1$ where, from Fig. 2b, damped values of l occur. Much of this variation may be because of inaccuracies in the present method of obtaining l but the results do indicate that $A^* = 26$ is somewhat of an average value, at least for the lower curve, and that evaluating the τ, ρ, and μ at the wall is perhaps a better method than any of the others shown.

In conclusion, turbulent shear stress and eddy viscosity variations have been obtained from velocity and density profile data at Mach 20.8 and $T_w/T_t = 1.0$. The results indicate that in the outer region an eddy viscosity expression defined as $\epsilon/(u_e\delta_i^*)$ and the mixing length are appreciably greater than those previously obtained for $M_e \leq 5$, probably because of low Reynolds number effects. In the wall region the results suggest a mixing length expression with a slope of 0.6; a value again typical of low Reynolds number turbulent boundary layers. The data from the thick sublayer present at this Mach number allowed a preliminary investigation of the wall damping effect upon mixing length. The results indicate that evaluating the damping scale using wall conditions is valid.

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